am TAHA AMEEN , ID @00066555 Taha Ameen

Exam I: MTH 420, Spring 2018

Ayman Badawi

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QUESTION 1. (i) Give me an example of a finite commutative ring A such that $char(A) = 11$, but A is not an integral domain.

Consider $\mathbb{Z}_{11} (+) \mathbb{Z}_{11}$ (FINITE). Then: "1" $= (1, 0)$ and $|(1,0)| = 11$.

$\therefore char(\mathbb{Z}_{11} (+) \mathbb{Z}_{11}) = 11$. BUT: $(0,1) * (0,1) = (0,0)$. Hence, it is NOT an I.D

(ii) Give me an example of a UFD, say A, that is a GCD-domain, but for some $a, b \in A$, we cannot write $gcd(a, b)$ as a linear combination of a and b.

$\mathbb{Z}[x]$ is a UFD and hence a gCD domain.

Let $a = 3, b = x$. $gcd(3, x) = 1$ But $1 \neq 3c_1 + xc_2$ for any $c_1, c_2 \in \mathbb{Z}[x]$

(iii) Let A be a finite commutative ring. Convince me that $N(A) = J(A)$.

In Finite Rings: Prime Ideals and Maximal Ideals are the Same

$\therefore \bigcap_{\forall i}$ (All Prime Ideals) $= \bigcap$ (All Maximal Ideals)

$\therefore \quad N(A) = J(A)$

(iv) Give me an example of an integral domain D such that D has an irreducible element, say d, but d is not a prime element of D.

LET $D = \mathbb{Z}[x^2, x^3]$. $d = x^2$ is Irreducible

But $\exists\ x^6 \in \mathbb{Z}[x^2, x^3]$ s.t. $x^6 = x^3 * x^3$

But $x^2 \nmid x^3$. $\therefore x^2$ is NOT Prime

QUESTION 2. (i) let I be a proper primary ideal of a commutative ring A, and let $F = \{x \in A \mid x^n \in I\}$. Prove that F is a prime ideal of A.

$F = \{x \in A \mid x^n \in I\}$. To Show: F is prime.

WE SHOW: whenever $ab \in F$, then $a \in F$ OR $b \in F$.

Let $ab \in F$. $\therefore (ab)^n \in I$. | By Definition

$\Longrightarrow a^n b^n \in I$ | $\because$ Ring Commutes

$\therefore a^n \in I$ OR $(b^n)^m \in I$ | $\because$ Ring is Primary

i.e. $b^{nm} \in I$

$\Downarrow$

$a \in F$ (OR) $b \in F$. $\therefore F$ is prime ∎

(ii) Let I be a proper ideal of a commutative ring A. Prove that I is a prime ideal of A if and only if the set $S = R - I$ is a multiplicatively closed subset of A.

(1) Let I be a prime Ideal. To Show: $S = R \setminus I$ is Multiplicatively Closed

DENY. $\therefore \exists\ d_1, d_2 \in S$ s.t $d_1 d_2 \notin S$.

Equivalently: $\exists\ d_1, d_2 \notin I$ s.t. $d_1 d_2 \in I$ $\quad | \because I = R \setminus S$.

But I is a prime Ideal.

Contradiction.

(2): Let $S = R \setminus I$ is Multiplicatively Closed. To Show: I is a prime Ideal.

DENY. $\therefore$ I is an IDEAL of R, but NOT prime.

Then $\exists\ d_1, d_2 \in I$ s.t. $d_1 \notin I$ and $d_2 \notin I$.

$\therefore\ d_1 \in S$ and $d_2 \in S$.

$\therefore\ d_1 d_2 \in S \quad | \because S$ is Multiplicatively closed.

$\therefore\ d_1 d_2 \in S \cap I$. But $S = R \setminus I$

and $I \cap R \setminus I = \phi$ Always.

Contradiction.

(iii) Briefly, how will you construct an integral domain A with exactly one nonzero maximal ideal, say M, such that $Z \subset A \subset Q$? What is $J(A)$? What is N(A)? are they equal? No

Let p be a prime natural number. (say, $p = 2$).

Let $A := \left\{ \frac{a}{b} \,\middle|\, p \nmid b,\ a \in Q,\ b \in Q^* \right\}$.

$Z \subset A \subset Q \quad | \because b = 1$ is Allowed

Then pA is the ONLY Maximal Ideal of A.

$\therefore\ J(A) = \bigcap_{\forall i} M_i \implies J(A) = pA$.

Also, $\therefore\ pA$ is also a prime Ideal.

Note {0} and pA are the only prime ideals of A.
N(A) = {0}. Hence N(A) not equal J(A)

QUESTION 3. (i) Convince me that $A = Z_3 \times Z_5 \times Z_3$ is not ring-isomorphic to $B = Z_{45}$ (Hint: Find $|U(A)|$ and $|U(B)|$)

$|U(A)| = (2)(4)(2) = 16$ | $\because \mathbb{Z}_3$ and $\mathbb{Z}_5$ are fields.

$|U(B)| = \phi(45) = 6(4) = 24$ | $\because 45 = 3^2 \times 5$

$\therefore$ THEY Cannot be Isomorphic $(3-1)\cdot 3 \cdot (5-1)\cdot 5^0$

(ii) Let $f(x) = x^3 + 2x + 1 \in A = Z_3[x]$, and $I = f(x)A = (f(x)) = span\{f(x)\}$. Convince me that $F = A/I$ is a field. How many elements does F have? (just the number, do not list all elements)

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We Show: I is prime and F is finite.

$f(0) = 1,\ f(1) = 1,\ f(2) = 1$ $\therefore$ $f(x)$ is Irreducible

($\because f$ is Monic and degree 3 in an I.D. f has no roots in $\mathbb{Z}_3 \Rightarrow f$ is Irreducible in $\mathbb{Z}_3[x]$)

BUT: $\mathbb{Z}_3[x]$ is a PID (and hence a UFD) ($\because \mathbb{Z}_3$ is a field)

$\therefore$ f is Irreducible $\Rightarrow$ f is Prime $\Rightarrow$ (f) is a PRIME IDEAL

$\therefore$ $F = A/(f)$ is an Integral DOMAIN. [CONTD ON PREVIOUS PAGE]

(iii) Let $f : A \to B$ be a ring-homomorphism that is ONTO (A, B are commutative). Prove that $f(1_A) = 1_B$. Hence prove that $f(U(A))$ is a subgroup of $U(B)$.

Since f is ONTO, $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$.

To Show: $f(1_A) = 1_B$. DENY. $\therefore \exists a \in A$ s.t. $f(a) = 1_B$ and $a \neq 1_A$.

Since f is a ring homomorphism, OK

$f(a \cdot 1) = f(a) \cdot f(1_A) = 1_B$. $\therefore 1_B \cdot f(1_A) = 1_B \Rightarrow f(1_A) = 1_B$.

(Multiplying with 1_B^{-1}) [CONTD. ON PREVIOUS]

(iv) Let $f : A \to B$ be a ring-homomorphism, where A is a commutative ring and B is an integral domain such that $f(a) \neq 0$ for some $a \in A$. Prove that $f(1_A) = 1_B$, and hence prove that $f(U(A))$ is a subgroup of $U(B)$.

B is an Integral domain.

To Show: $f(1_A) = 1_B$.

DENY. $\therefore f(1_A) = b$ where $\boxed{b \neq 1_B}$.

$\exists a \in A$ s.t. $\boxed{f(a) = c \neq 0}$

consider $f(a) = f(a \cdot 1) = f(a) \cdot f(1) = c \cdot b$.

$\therefore$ $c = c \cdot b \Rightarrow c - cb = 0 \Rightarrow c(1-b) = 0$

Since B is an I.D.

$c = 0$ (OR) $1 - b = 0$, i.e. $b = 1_B$

In Both cases we have a contradiction.

$\therefore f(1_A) = 1_B \Rightarrow f(U(A)) < U(B)$ by previous Question.

QUESTION 4. (i) Let A be a commutative ring and $w \in N(A)$. Prove that $w + u \in U(A)$ for every $u \in U(A)$. (Can you give a simpler proof than the one that you gave in the HW?)

- $\forall u \in U(A) \wedge \forall m \in J(A)$, $u + m \in U(A)$.
- $N(A) \subseteq J(A)$.

$\therefore$ From above 2 statements

$w \in N(A) \Rightarrow w \in J(A)$

$\therefore \forall u \in U(A)$, $w + u \in U(A)$ ■

(ii) Let A be a commutative ring and M be a maximal ideal of A. Prove that $M[x]$ is never a maximal ideal of $A[x]$. (Hint: Construct a certain ring homomorphism that is onto)

To Prove: $M[x]$ is never a Maximal Ideal of $A[x]$.

Proof: $\phi : A[x] \longrightarrow \frac{A}{M}[x]$.

$\phi(a_n x^n + \ldots + a_1 x + a_0) \rightarrow (a_n + M)x^n + \ldots + (a_1 + M)x + (a_0 + M)$

is a ring Homomorphism that is ONTO. (Trivial)

Clearly, $Ker(\phi) = M[x]$ | $\because$ when $a_i \in M \; \forall i$, the Image is M.

$\therefore \frac{A[x]}{M[x]} \approx \frac{A}{M}[x]$.

$F[x]$ is a PID.

$\frac{A[x]}{M[x]}$ is a PID

[Contd. on Previous Page].

(iii) Let $A = Z_3[x]$, $f(x) = x^2 + x \in A$ and $I = (x^2 + x) = \text{span}\{x^2 + x\}$. Prove that A/I is ring-isomorphism to $Z_3 \times Z_3$ (note xA and (x+1)A are prime ideals (maybe maximal ideals too!))

$f(x) = x^2 + x = x(x+1)$ and $I = (x(x+1))$

Let $I_1 = xA$ and $I_2 = (x+1)A$.

I_1 and I_2 are Coprime.

$(\because \exists -x \in I_1$ and $x+1 \in I_2$ s.t. $-x + x + 1 = 1)$

$\therefore$ By the (CHINESE REMAINDER THEOREM)

$$\frac{\mathbb{Z}_3[x]}{I_1 \cap I_2} \approx \frac{\mathbb{Z}_3[x]}{I_1} \times \frac{\mathbb{Z}_3[x]}{I_2}.$$

Clearly, $I = I_1 \cap I_2$

PTO.

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com

Answer 3 (ii): (contd.).

F is an Integral domain and F is Finite

$\therefore$ F is a FIELD.

Claim: F has 27 Elements.

PROOF:

Since $\mathbb{Z}_3[x]$ is Euclidean ($\because \mathbb{Z}_3$, is a field)

$a \in \mathbb{Z}_3[x] \Rightarrow a = bq + r$.

By quotienting out by $p(x)$,

it is clear that

$$g \in \frac{\mathbb{Z}_3[x]}{I} \Rightarrow g = a_2 x^2 + a_1 x + a_0 + I$$

and there are 3 choices for a_0, a_1, a_2

$$\therefore |F| = 3^3 = 27$$

Answer 3 (iii): $f(1_A) = 1_B$ is a contradiction.

$\therefore f(1_A) \neq 1_B$. To show: $f(U(A)) < U(B)$

$u \in U(A) \Rightarrow f(1_A) = f(u * u^{-1}) = f(u) * f(u^{-1}) = 1_B$.

$\therefore u \in U(A) \Rightarrow f(u) \in U(B)$. Also, $f(U(A))$ is CLOSED ($\because f(u_1) * f(u_2) = f(u_1 u_2)$)

$\therefore f(U(A)) < U(B)$.

Answer 4 (ii) (Contd...)

$$\frac{A[x]}{M[x]} \approx \frac{A}{M}[x]$$

and $\frac{A}{M}$ is a Field

$\therefore \frac{A}{M}[x]$ is an Euclidean domain ($\Rightarrow$ PID).

We show:

$\frac{A}{M}[x]$ cannot be a field.

DENY $\therefore \frac{A}{M}[x]$ is a field.

But Consider $\alpha(x) = (1+M)x + (a_0+M) \in \frac{A}{M}[x]$

Clearly $\alpha^{-1}(x)$ does NOT Exist

($\because$ $(1+M)$ is NOT Nilpotent, even if (a_0+M) was a unit).

Contradiction

$\therefore \frac{A}{M}[x]$ is NOT a field

$\Downarrow$

$\frac{A[x]}{M[x]}$ is NOT a field

$\Downarrow$

$M[x]$ is NOT Maximal.

■

To Prove:

$$\frac{\mathbb{Z}_3[x]}{(x)} \approx \frac{\mathbb{Z}_3[x]}{(x+1)} \approx \mathbb{Z}_3 .$$

Define:

$\rightarrow$ $\phi_1 : \mathbb{Z}_3[x] \longrightarrow \mathbb{Z}_3$

s.t. $\phi_1(f(x)) = f(0)$

$\left\{ \begin{array}{l} \phi_1(f_1(x)+f_2(x)) = f_1(0)+f_2(0) = \phi(f_1)+\phi(f_2) \\ \text{and } \phi_1(f_1(x)\cdot f_2(x)) = f_1(0)\cdot f_2(0) = \phi(f_1)\phi(f_2) \end{array} \right.$

This is a ring homomorphism

It is ONTO.

$(\because \forall\, a \in \mathbb{Z}_3 \quad \exists\ x+a \in \mathbb{Z}_3[x] \text{ s.t. } \phi(x+a) = a)$

$\therefore \dfrac{\mathbb{Z}_3[x]}{Ker(f)} \approx \mathbb{Z}_3 .$ Here: $Ker(f) = (x)$

$(\because (x) = \{x\,l(x) \mid l(x) \in \mathbb{Z}_3[x]\}$

and $\phi((x)) = 0 * l(0) = 0)$.

Also, $\phi(m(x)) \neq 0$ if $m(0) \neq 0$

i.e. $m(x)$ has a constant term.

$\rightarrow$ $\phi_2 : \mathbb{Z}_3[x] \longrightarrow \mathbb{Z}_3$

s.t. $\phi_2(f(x)) = f(2)$.

This is a ring homomorphism (same as above)

It is ONTO $(\because \forall\, a \in \mathbb{Z}_3 \ \exists\ g(x) = (x+1)+a$ s.t.

$\phi(g(x)) = g(2) = a)$

$\therefore \mathbb{Z}_3[x]/Ker(f) \approx \mathbb{Z}_3 .$ Here: $Ker(f) = (x+1)\mathbb{Z}_3[x]$.

$(\because l(x) = (x+1)*m(x) \Longrightarrow \phi(l(x)) = l(2) = (2+1)*m(2) = 0$.

AND $l(x) \notin (x+1)\mathbb{Z}_3[x] \Longrightarrow \phi(l(x)) \neq 0.)$.

$\therefore \dfrac{\mathbb{Z}_3[x]}{I} \approx \mathbb{Z}_3 \times \mathbb{Z}_3$